

Value and Information: A Profit Maximizing Strategy for Maxwell's Demon

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Applications of Lagrange's method of undetermined multipliers in physics and economics suggest some analogies between the two disciplines. Applying Jaynes' approach to information theory reveals the long standing problem of Maxwell's demon in thermodynamics as a constrained optimization problem where information and energy are dual concepts. Temperature emerges as the energetic price of information. This approach highlights the economic features of the Maxwell demon discussed by Bridgman. It also suggests that an analog to temperature (or the price of information) should be present in any optimization problem constrained by costly information. A trivial economics example is presented where generalized temperature measures informational efficiency in the sense of Grossman and Stiglitz. © 1995 John Wiley & Sons, Inc.

The joint view of thermodynamics and economics that motivates this article goes back to Sadi Carnot's [4] initial purpose in considering the efficiency of heat engines. Percy Bridgman [2], Nobel laureate in 1946 for his work on materials at high temperatures and pressures, had this view in mind in commenting on the second law:

It must be admitted, I think, that the laws of thermodynamics have a different feel from most of the other laws of the physics. There is something more palpably verbal about them—they smell more of their human origin. The guiding motif is strange to most of physics: namely, a capitalizing of the universal failure of human beings to construct perpetual motion machines of either the first or the second kind. Why should we expect

nature to be interested either positively or negatively in the purposes of human beings, particularly purposes of such an unblushingly economic tinge?

Two principal applications of Lagrange's method of undetermined multipliers occur in physics and in economics. In economics, these undetermined multipliers are usually related to prices. Building on the framework of E.T. Jaynes [13], we can treat thermodynamic systems in terms of constrained optimization and derive temperature as an undetermined multiplier. From this viewpoint, and considering an analogy to economics, it becomes apparent that temperature is related to the unit cost of information in an equilibrium system. This approach provides a new perspective on the paradox of the Maxwell demon that deals directly with the economic features

discussed by Bridgman. It also points to new ways of modeling the price of information in economic systems, a problem of considerable importance which has thus far eluded a complete resolution.

The sorting demon introduced by Maxwell [15] was able to violate the statement of the second law of thermodynamics which precludes continuous extraction of work from heat (perpetual motion machines of the second kind). This was to be accomplished by selecting molecules at a barrier between two compartments of gas and allowing high speed transit in one direction only. The sorting collects high speed molecules on one side of the barrier and low speed molecules on the other, thus creating a temperature difference to drive a heat engine. The violation assumes that the demon can get information about the velocity of molecules without paying any price for that information.

Jaynes' approach starts by regarding statistical mechanics as a form of statistical inference rather than as a physical theory. This allows him to distinguish sharply between the physical and statistical aspects of the subject. The former consists of the correct enumeration of the n possible states of a system (indexed by i) and their properties; the latter is a straightforward example of statistical inference.

The Jaynes analysis then focuses on the expression $\sum f_i \ln(f_i)$, where f_i is the probability that an equilibrium system is in its i th state. This function had been previously introduced both in the entropy measure and in the information measure. In the latter, it can be interpreted as the average information gain or "surprise" experienced when the actual state of the system is determined. This follows from Shannon's [21] argument that $\log_2(f_i)$ represents the information gained (measured in units called *bits*) in learning the system is in state i . We deviate slightly from the usual formalism and use I for average information, employing natural rather than binary logarithms. This means we measure information (in units called *ebits*) as

$$I = -\sum_{i=1}^n f_i \ln(f_i). \quad (1)$$

The quantity I can be calculated for any discrete normalized probability function and measures the average information we would have if we knew the state of the system. It is a measure of uncertainty, analogous but not equivalent to variance. We reserve the term "entropy" for the function introduced in statistical mechanics:

$$S = -k \sum_{i=1}^n f_i \ln(f_i) \quad (2)$$

where the probabilities refer to the quantum states of thermodynamic systems at equilibrium and k is Boltzmann's constant. Thus in equilibrium statistical mechanics

$$S = kI \quad (3)$$

and entropy is represented by a specific application of the statistical measure I . Entropy is k times the average information one would gain upon learning the precise microstate of a thermodynamic system at equilibrium, given the macroscopic parameters such as temperature and average energy that define the f_i 's. Entropy is a measure of our ignorance of the quantum state of an equilibrium system once the defining macroscopic variables are known.

Jaynes proposes a general method of inference that is an extension of Laplace's *principle of insufficient reason*. He asserts that wherever we make inferences on the basis of partial information we choose the distribution that maximizes entropy or average information gain relative to what is known: "The maximum-entropy distribution may be asserted for the positive reason that it is uniquely determined as the one which is maximally noncommittal with regard to missing information."

We can now move on to make contact with the standard approach to statistical mechanics. Suppose we have a system with average energy E . The system may be in any one of n different quantum states. The probability that the system is in the i th such microstate is f_i and the quantized energy of the system in that state is E_i . We wish to follow the approach of Jaynes and make the most noncommittal statement possible about the state of the system. Therefore we maximize the expected information gain from knowing the actual state of the system,

$$\max_{\{f_i\}} \sum_{i=1}^n f_i \log(f_i). \quad (4)$$

This maximization is equivalent to the Gibbsian procedure of maximizing the entropy of Eq. (2) which he introduced in a postulational way. The maximization must be carried out subject to the constraint that the average energy or internal energy of the system is a constant, E ,

$$\sum_{i=1}^n f_i E_i = E \quad (5)$$

and is subject to the normalization constraint that the probabilities of the n states add to one

$$\sum_{i=1}^n f_i = 1. \quad (6)$$

A procedure for solving this system is to apply Lagrange's method of undetermined multipliers in integral form: the method of constrained optimization (see Weinstock [22]). Although much of physics can be described by constrained optimization principles such as the principle of least action, Fermat's principle of least time, etc., modern physicists seem to prefer a reduced form statement of these problems because they are suspicious of what Margenau [14] calls "the spurious character of purpose injected into nature." However, in economics, the integral form of Lagrange's method is preferred

because first principles such as profit maximization have a clear teleological content. To facilitate a comparison of the use of Lagrange's method in the two disciplines, we state the problem in integral form:

$$\mathcal{L} = - \max_{\{f_i\}} \sum_{i=1}^n f_i \ln(f_i) - \alpha \left[\sum_{i=1}^n f_i - 1 \right] - \beta \left[\sum_{i=1}^n f_i E_i - E \right] \quad (7)$$

To solve for the constrained maximum, differentiate \mathcal{L} with respect to f_i through f_n and with respect to the undetermined multipliers α and β . The first order conditions are:

$$0 = \ln(f_i) + 1 + \alpha + \beta E_i \quad i = 1, \dots, n \quad (8)$$

$$0 = \sum_{i=1}^n f_i - 1 \quad (9)$$

$$0 = \sum_{i=1}^n f_i E_i - E. \quad (10)$$

In principle the $n + 2$ Eqs. (8), (9), and (10) could be solved to recover the $n + 2$ unknowns ($f_1, \dots, f_n, \alpha, \beta$). We need not do this explicitly. Rather we begin by rewriting Eq. (8) as

$$f_i = e^{-(\alpha + \beta E_i + 1)} \quad i = 1, \dots, n \quad (11)$$

By substituting Eq. (11) into Eq. (9), we can obtain an expression for α in terms of β and thus eliminate α . Thus,

$$f_i = \frac{e^{-\beta E_i}}{\sum_{j=1}^n e^{-\beta E_j}}. \quad (12)$$

To evaluate f_i we must solve for β in terms of macroscopic variables. For this we must refer to thermodynamics.

Consider a single homogeneous substance at or very near equilibrium and in contact with an isothermal reservoir (Maxwell's demon is assumed to operate in such an environment). The first law of thermodynamics states that for such a system

$$dE = dQ - dW. \quad (13)$$

The change of energy of the system, dE , is equal to the energy that enters as heat, dQ , minus the energy that leaves as work, dW . For very slow quasi-static transformations of the kind dealt with in equilibrium thermodynamics, the heat entering the system can be replaced by the term TdS , the temperature times the entropy change. Indeed for classical equilibrium thermodynamics entropy is defined by the relation $dS = dQ/T$ for quasi-static transformations. An engine receives energy from an external source ($dE > 0$) and performs work ($-dW > 0$), producing heat as a by-product ($TdS > 0$).

The energy change of the system can also be obtained by differentiating Eq. (5) totally:

$$dE = \sum_{i=1}^n E_i df_i + \sum_{i=1}^n f_i dE_i. \quad (14)$$

The first term in Eq. (14) describes changes in the probabilistic microstructure of the system and can be equated to TdS . The second term in Eq. (14) describes changes in the energy associated with each microstate, and can be equated to $-dW$. After equating terms in Eq. (13) and Eq. (14), substituting into Eq. (12) and simplifying, by an involved argument utilizing the constructs that we have already introduced it can be shown (Hill [11]) that

$$\beta = \frac{1}{kT}. \quad (15)$$

The Maxwell-Boltzmann probability function (which gives the probabilities of the possible quantum microstates of the system in terms of the macroscopic parameter T and the E_i 's, and is one of the fundamental concepts underlying statistical mechanics) is obtained by substituting Eq. (15) into Eq. (12). For our purposes, however, it is more important to note that the undetermined multiplier β has the dimensions of ebits per unit of energy and is proportional to the inverse of the absolute temperature of the system.

The duality theorem of constrained optimization (see Samuelson [18]) implies that, in this problem, extremizing information subject to an energy constraint and extremizing energy subject to an information constraint generate equivalent first order conditions. However the undetermined multipliers in one version of the problem are the reciprocals of those in the other. Thus an alternative derivation of the Maxwell-Boltzmann distribution would result in an undetermined multiplier directly proportional to temperature.

In economics, undetermined multipliers are usually related to prices. Does this indicate that temperature can be thought of as somehow defining a relative price governing a tradeoff between energy and information (or entropy)?

To investigate this, we turn now to the use of Lagrange's method of undetermined multipliers in the standard economic model for a profit maximizing competitive firm (see, e.g., Samuelson [18]).

Let us imagine that a Maxwell's demon is operating such a firm. The firm converts heat into work using a container of gas initially in contact with an infinite isothermal reservoir. The container is isolated from the reservoir and divided into two chambers by a wall with a door which the demon operates, allowing only fast moving molecules to move into the left chamber and slow moving molecules to move into the right chamber. Note that this process reduces the entropy of the system and thus apparently violates the second law of thermodynamics.

It also creates a temperature difference between the two chambers that is used to do work. Heat is allowed to flow from the left to the right chamber through a Carnot engine, performing external work until the temperatures of the two chambers equalize at a lower value. The chambers are then placed in contact with the reservoir again and heat flows in, raising

the temperature and energy content of the chambers to the original value. The process is then repeated. The net result is to extract heat from the reservoir and do external work.

Note that to operate the door, the demon must collect information I about the micro-state of the physical system, i.e., which molecules are moving at what speed. This decreases the entropy of the gas. To collect this information the demon must pay the cost of operating a system of molecular motion detectors, and this cost preserves the second law of thermodynamics.

Define p_I as the cost the demon pays per ebit of information. Let us assume the demon can sell the work energy it produces in a perfectly competitive market at price p_w per Joule. It cannot alter these prices; they are determined by economic and physical constraints external to its firm. The demon's profit π is

$$\pi = p_w W - p_I I. \quad (16)$$

Let us postulate what economists call a production function W to describe the quantity of work energy (output) produced when the firm uses a given quantity of information (input):

$$W = W(I). \quad (17)$$

The function W may be thought of as describing the workings of the physical system of molecular motion detectors, levers, Carnot engines, etc., the demon employs. We consider the nature of W below.

The demon wishes to obtain the maximum profit possible given the physical production system W . Its problem is therefore to maximize the profits given by Eq. (16) subject to the operating restrictions expressed by the production function (17). To solve this problem, we use the standard formalism of microeconomics and apply Lagrange's method of undetermined multipliers in integral form, obtaining the expression

$$\mathcal{L} = p_w W - p_I I - \lambda [W - W(I)]. \quad (18)$$

To obtain the maximum, we solve for the first order conditions with respect to W , I and the undetermined multiplier λ . They are

$$P_w = \lambda \quad (19)$$

$$P_I = \lambda \frac{dW}{dI} \quad (20)$$

along with the production function (17).

Eq. (19) states that the Lagrange multiplier equals the price of energy. Eq. (20), says that undetermined Lagrangian multiplier is also equal to the cost of a unit of information (p_I) times

the number of units of information required to produce an additional unit of work (dI/dW). Thus λ is the demon's marginal cost of production: the cost of the information needed to produce a unit of energy on the margin. By equating price with marginal cost, we combine Eq. (19) and Eq. (20) to eliminate λ and obtain the standard economic equation defining profit maximization. (See, e.g., Samuelson [18]).

$$\frac{p_I}{p_w} = \frac{dW}{dI}. \quad (21)$$

With prices determined externally, Eq. (22) and the production function Eq. (17) can be solved jointly to determine the demon's profit maximizing operating strategy just as Eqs. (8), (9), and (10) could have been jointly solved to find the unknowns of the physics problem described above.

For our purposes, however, it is again of more interest to consider characteristics of the system implied by a solution. Of particular interest is the implication that the unit cost of information relative to the price of work energy, p_I/p_w , should be equal to the physical quantity of energy produced by a unit of information on the margin, dW/dI .

In order to understand the terms of this information energy tradeoff implicit in dW/dI we must understand the physical constraints that limit the form of the demon's production function. Let us suppose that the demon's production process cannot violate the first law of thermodynamics, Eq. (13) which we restate here for a system consisting of the two chambers and the Carnot engine:

$$dE = TdS - dW. \quad (22)$$

Since the system is operated in a cycle, it converts heat from the reservoir into external work with no net change in the internal energy of the system ($dE = 0$). Thus, substituting Eq. (3), which related entropy to information, into Eq. (22) with dE equal to zero and differentiating yields,

$$\frac{dW}{dI} = kT \quad (23)$$

where k is Boltzmann's constant and T is absolute temperature.

Thus, in an *ideal* demonic system the energy produced per marginal unit of information is kT . If the demon's firm is inefficient in any way, its energy production per unit of information would be lower than this. Thus Eq. (22) implies that for perfectly efficient demons, price equals marginal cost when

$$\frac{p_I}{p_w} = kT. \quad (24)$$

Note that p_I is in \$/ebit and p_w is in \$/joule so p_I/p_w is in joules per ebit which is the appropriate units and can be thought of as the unit cost of information denominated in Joules rather than dollars.

Of course, prices are what they are. If the energy-information tradeoff is more favorable than kT Joules per ebit the demon should operate at infinite capacity to make infinite profit. If the tradeoff is kT Joules per ebit or less, there is no profit in being a Maxwell's demon.

Thus the second law of thermodynamics in this context is equivalent to the statement that *the unit cost of information (denominated in Joules per ebit) about the microstructure of the system must be at least kT .*

Any demon operating within the standard paradigm of microeconomics and constrained by the laws of equilibrium statistical mechanics must be paying a cost equivalent to the value of kT Joules of work energy for each ebit of information. At most, she can be breaking even by exploiting an information energy tradeoff defined by the temperature of the system. The demon cannot be costlessly extracting information about the microstructure of the system and using it to do work.

Indeed, Szillard [19] obtained this result by assuming the second law of thermodynamics and showing that a demon must increase entropy or the law can be violated. Brillouin [3] derived the same result by focusing on the process of measurement and showing in terms of signal to noise that there is a price to the measurement.

The discussion given above does indicate, following Bridgman, an analogical relationship between the "no free lunch" dictum of economics and the second law of thermodynamics. A more conservative approach to the preceding sentence is to replace analogical by metaphorical.

These considerations suggest a new approach to pricing information in economics. A gigantic literature has developed since Stigler's [20] original work on the economics of information. Various measures of lack of information, risk or uncertainty have been used; but it is fair to say that a uniform way of treating information directly as a commodity with a price has not emerged.

We speculate that this is because the so-called "marginalist" economists such as Walras and Jevons who laid the foundations of modern economics in the 1870s largely ignored information costs. As Mirowski [16] shows, these economists developed much of the currently accepted structure of economic theory by borrowing from 19th-century physics.¹ Standard economic tools such as utility maximization, profit maximization, budget constraints and equilibrium are based explicitly on analogies with the physics of the day. Jaynes' [13] approach, which clarified the role of information in thermodynamic systems was of course not understood at the time of this cross fertilization. Thus concepts such as the price of information or the production of information must be grafted on to our "classical" picture of the consumer and the firm.

Economists should be aware that the concept of temperature provides a uniform way of treating information as a quan-

tity with a price in thermodynamics. Thermodynamic free energy transformation functions have a TdS term indicating the energetic value of missing information. Entropy (missing information) and temperatures always occur as dual variables in thermodynamics. Energy, which corresponds to value in the tradeoff resulting from the Lagrangian analysis, always involves a product of these two variables.

Being able to characterize information directly as an economic good with a unit price, rather than indirectly as information sets on which conditional moments are formed, would be useful in a number of applications in economics. Our analysis suggests that this requires looking for temperature analogs in economic systems that one wishes to characterize by conventional information measures. For example, Palmer [17] by *fiat* introduces a "pseudo-temperature" in order to apply optimization techniques from statistical mechanics to the travelling salesman problem and other complex optimization problems. Below we outline a rough analogy between Maxwell's demon and a financial entrepreneur based on the fact that both seek to profit by using information unavailable to others.

In modeling the flow of information in financial markets, the concept of *generalized temperature* as the relative price of information could be of interest. If information arrives spontaneously and is simultaneously available to all market participants, as models of informationally efficient financial markets assume (see, e.g., Fama [9]; Brealey, Myers, and Marcus, [1]), the *generalized temperature* of the system is zero. Grossman and Stiglitz [10] can then be taken as showing that generalized temperatures of financial markets must be positive.

If an entrepreneur is able to acquire new information² I at price p_i per ebit and use it to make D trades each yielding a sure gain of p_D , and if the number of profitable trades it can make is related to the quantity of private information it has by a production function $D = D(I)$, then its profit maximization problem is to optimize:

$$L = p_D D - p_i I - \lambda [D - D(I)] \quad (25)$$

where λ is an undetermined multiplier.

In practice traders cannot make infinite profits off a piece of new information, even though many economic models assume this. The size of a trader's position is limited by her risk aversion and by institutional features of the economy like credit limits (see, e.g., DeLong et al. [6]). More fundamentally, it is also limited by the inevitable eventual publicizing of private information, and by the fact that other traders can infer the nature of private information by observing each other's trades (see Grossman and Stiglitz [10]). This negative feedback makes the production function D convex, that is $dD/dI > 0$ and $d^2D/dI^2 < 0$.

The same analysis that was used to solve for a profit maxi-

mization strategy for Maxwell's demon can be applied here. The profit maximization condition is that at the margin,

$$\frac{p_I}{p_D} = \frac{dD}{dI} \quad (26)$$

If D is sufficiently convex, a solution to Eq. (26) exists.

We define the *generalized temperature* τ as the value of a marginal ebit of information [$\tau = (dD/dI)p_D$ which has units of \$/ebit]. The *generalized temperature* of the market constructed in this fashion is a measure of the availability of opportunities for arbitrage. *Generalized temperature* is thus a measure of the extent to which markets are not informationally efficient in the sense of Grossman and Stiglitz [10]. Information is incorporated in prices only up to the point where its marginal cost equals its marginal value, which in turn equals the generalized temperature of the market. This overall viewpoint relates to expressions used by traders such as "*the market is hot*" to describe a large volume of trading presumably due to private information. A further analogy can be drawn between informationally efficient markets and the third law of thermodynamics, which states that, at absolute zero temperature, entropy (or missing information) is zero.

In conclusion, the formalisms shared by thermodynamics and economics allow one to think of temperature as the relative price of information in an equilibrium system. From this heuristic viewpoint, it is possible to develop a new perspective on the paradox of Maxwell's demon that includes the economic features discussed by Bridgman [2]. In turn, this approach leads to the concept of a *generalized temperature* in any system where a quantity is being extremized subject to a costly information constraint.

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FOOTNOTES

1. For example, in his *Mathematical Psychics*, Edgeworth [8] spends much of chapter one on the importance of analogies from physics in his formulation of economics. Note that Edgeworth [7] contains the first use of a Lagrangian multiplier in economics of which we are aware. The personal correspondence of Léon Walras (see Jaffé [12]), especially that with Boninsegni and Poincaré, contains much discussion of analogies between classical mechanics and Walras' views about economics. See Mirowski [16] for an excellent overview of economic's extensive borrowing from 19th-century physics.
2. In the following discussion, information can be thought of as an abstract good, like "capital" in standard microeconomics. A more concrete approach to which this analysis is relevant is Cox, Ross, and Rubenstein [5] in which stock prices go up $u\%$ with probability q at any given time and down $d\%$ with probability $(1-q)$. This binomial process converges to the usual brownian motion stock process. An investor who knows the direction a stock price will move at a particular time gains no new information by observing that price change. On the other hand, an investor who has no such foreknowledge, using Shannon's [21] formulation, gains

$$I = -\sum_{i=1}^2 f_i \log_2(f_i) = -q \log_2(q) - (1-q) \log_2(1-q)$$

bits of information upon learning the actual movement of the price. Thus, if q is .5, the informed investor has precisely one bit of information everyone else has yet to learn.